

# Fractal Dimension of 3-Blocks in 4d, 5d, and 6d Percolation Systems

Gerald Paul<sup>1,\*</sup> and H. Eugene Stanley<sup>1</sup>

<sup>1</sup>*Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215*

Using Monte Carlo simulations we study the distributions of the 3-block mass  $N_3$  in 4d, 5d, and 6d percolation systems. Because the probability of creating large 3-blocks in these dimensions is very small, we use a “go with the winners” method of statistical enhancement to simulate configurations having probability as small as  $10^{-30}$ . In earlier work, the fractal dimensions of 3-blocks,  $d_3$ , in 2d and 3d were found to be  $1.20 \pm 0.1$  and  $1.15 \pm 0.1$ , respectively, consistent with the possibility that the fractal dimension might be the same in all dimensions. We find that the fractal dimension of 3-blocks decreases rapidly in higher dimensions, and estimate  $d_3 = 0.7 \pm 0.2$  (4d) and  $0.5 \pm 0.2$  (5d). At the upper critical dimension of percolation,  $d_c = 6$ , our simulations are consistent with  $d_3 = 0$  with logarithmic corrections to power-law scaling.

PACS numbers:

## I. INTRODUCTION

Percolation is a classic model for disorder [1, 2, 3]. Recently it has been recognized for bond percolation that clusters and blobs are the  $k = 1$  and  $k = 2$  cases of  $k$ -connected graphs ( $k$ -blocks), graphs in which all vertices are connected to every other vertex in the  $k$ -block by at least  $k$  independent paths [4, 5]. The fractal dimension  $d_3$  of 3-blocks in 2- and 3-dimensional percolation systems at the percolation threshold were found to be  $1.20 \pm 0.1$  and  $1.15 \pm 0.1$ , respectively [4].

The fact that the fractal dimensions are identical within error bars is consistent with the possibility that  $d_3$  might be independent of dimension. The focus of this paper is to determine  $d_3$  for  $d = 4, 5$ , and 6 using Monte Carlo simulations.

In the next section we study  $d_3$  for percolation on the Cayley tree in order to gain insight into the behavior of 3-blocks in very high dimension. In Section III we discuss the methods we use to generate large 3-blocks in 4d, 5d, and 6d. In Section IV we discuss our results.

## II. CAYLEY TREE RESULTS

Percolation on the Cayley tree has been used as a model for percolation for  $d \geq 6$ , the upper critical dimension of percolation. The cluster fractal dimension and blob fractal dimension, as well as a number of other critical exponents on the Cayley tree, are identical to those of percolation for  $d \geq 6$  [1, 2, 3]. Below we argue that for percolation on the Cayley tree  $d_k = 0$  for  $k \geq 3$  suggesting that while  $d_3$  may change little between  $d = 2$  and  $d = 3$ , eventually  $d_3$  decreases more rapidly approaching zero for  $d = 6$ .

To show that  $d_k = 0$  for  $k \geq 3$  for percolation on the

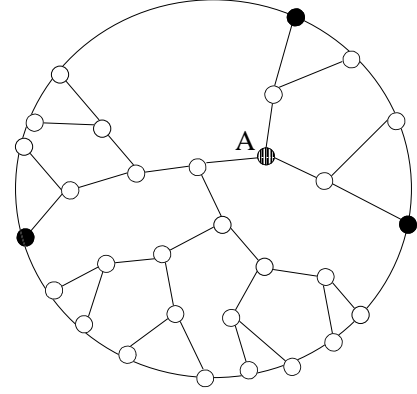


FIG. 1: Cayley tree in which each site except those on boundary has three neighbors. We note that even when all bonds are fully occupied, there can be only one site which is connected to three sites on the boundary. In this example the three sites on the boundary are the filled circles and the one site connected to them by independent paths is the striped circle denoted A.

Cayley tree, we make use of the concept of  $k$ -bone. Reference [4] generalizes the concept of backbone by defining a  $k$ -bone as the set of all sites connected to  $k$  disjoint sets of points by  $k$  independent paths. Thus clusters and backbones are  $k$ -bones with  $k = 1$  and 2 respectively. For a given  $k$ , the fractal dimension of  $k$ -bones and  $k$ -blocks are equal [4].

To see that for percolation on the Cayley tree the fractal dimension of a 3-bone is zero, we choose any 3 points on the boundary (Fig. 1) and observe that there is only one site which is connected to these points by independent paths. This result is independent of the size of the tree, even if the tree is fully populated. Hence the fractal dimension is zero. Clearly this argument holds for larger

\*Electronic address: gerrypp@bu.edu

$k$  and is meaningful as long as the branching factor in the Cayley tree is greater than or equal to  $k$ , and holds independent of size.

### III. SIMULATION METHOD

#### A. Statistical Enhancement Method

Randomly generated realizations in which large 3-blocks are present become more and more rare as the system dimension increases. In fact, if traditional techniques are used to generate realizations, for  $d$  as low as 4 the range of the values of the masses of 3-blocks created are so small that one cannot determine  $d_3$  either by finding the best collapse of plots of the distribution of masses or by inferring  $d_3$  from the slope of the power-law regime of the distributions.

To overcome this problem, we use a “go with the winners” method of statistical enhancement described in Ref. [6]. The basic idea of this approach in the context of a percolation cluster growth algorithm is as follows:

- (i) Before we start growing a cluster, assign a value of one to the weight  $W$  of the cluster.
- (ii) We use the Leath method to grow clusters [7]. While the cluster is growing, we calculate certain properties of the state of the cluster after every interval of  $n$  chemical shells of growth.
- (iii) If certain criteria on the properties of the state of the cluster that are described below are met, we “clone” the state so we have  $m$  copies (including the original) of the state, adjust  $W$  accordingly to  $W/m$  and continue growing each of these  $m$  clones. If these criteria are not met, simply continue growing the non-cloned cluster.

Cloning can take place multiple times during the growth of a cluster; the result is a tree structure of realizations where the leaves of the tree represent the completion of cluster growth.

Here  $m$  and  $n$  are parameters which can be tuned to achieve the desired level of “rareness” which can be reached. If  $n$  is large and  $m$  is small, there will be little cloning and we will generate clusters with weights only moderately smaller than without enhancement. If  $n$  is small and  $m$  is large, there will be much cloning and we will generate clusters with weights very much smaller than without enhancement. However, if  $n$  is sufficiently small and/or  $m$  is sufficiently large, cluster growth will effectively never end naturally, and we will not be able to extract useful information from the simulation.

From an implementation standpoint, it is not necessary to actually create copies of the state of the system in computer memory to create the clones; as noted in Ref. [6] we can effectively walk the clone tree in a “depth-first”

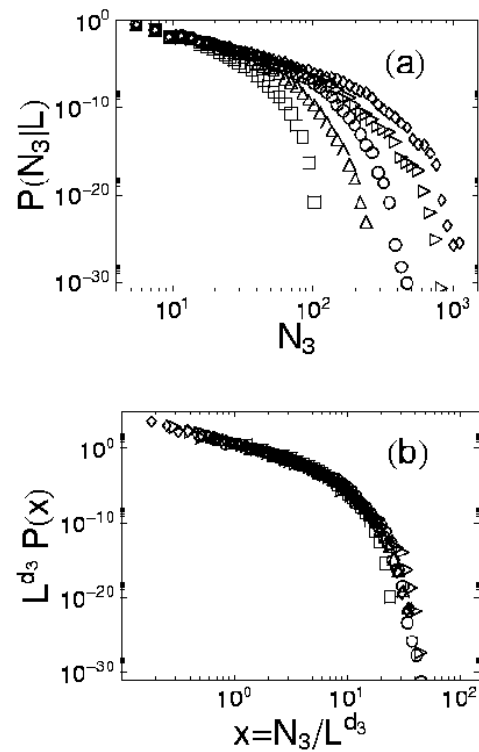


FIG. 2:  $P(N_3|L)$ , the distribution of 3-block mass  $N_3$  for (from left to right)  $L = 8, 16, 32, 64$ , and  $128$  for the case of four dimensions. (a) uncollapsed (b) collapsed using a value of  $d_3 = 0.7$ .

manner, completely treating a given clone before we begin treating the next clone. What is required is that we save the state of the system before we begin growth based on a clone so that we can return to that state when we begin growth on the next clone. This saving of state is accomplished naturally with a “last-in-first-out” stack in which we maintain information about sites in the cluster.

We first attempted to create realizations with large 3-blocks by creating very dense clusters. We set as our criteria for cloning the condition that the number of occupied bonds actually created during the  $n$  shell interval be larger than the number of times we determined whether a bond should be occupied times the bond occupation probability. While this algorithm is very effective in creating dense clusters, it did not result in large 3-blocks within the clusters. We were, however, successful in creating clusters with large 3-blocks by using a criterion which results in the creation of large blobs: clone if the most massive blob found in the cluster at the end of the interval is more massive than the largest blob created before growth in the interval is begun. That is, either an existing blob grows, one or more blobs merge or a new blob is created which is more massive than any existing previous to growth in the interval.

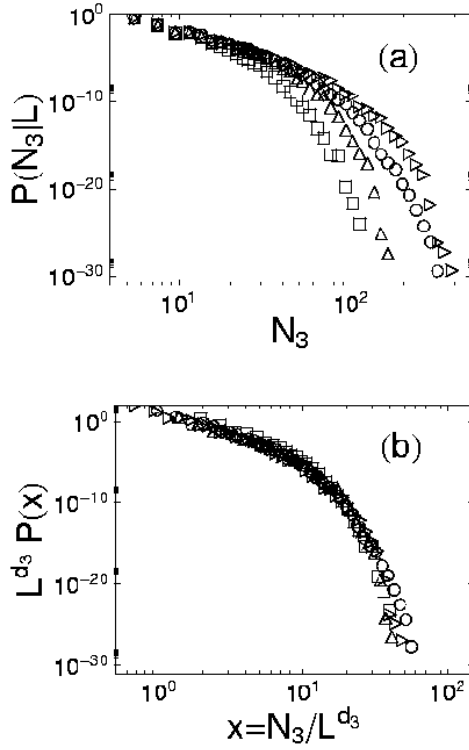


FIG. 3:  $P(N_3|L)$ , the distribution of 3-block mass  $N_3$  for (from left to right)  $L = 8, 16, 32$ , and  $64$  for the case of five dimensions. (a) uncollapsed (b) collapsed using a value of  $d_3 = 0.5$ .

### B. Incremental Cluster Decomposition

The decision whether to clone depends on a knowledge of the mass of the largest blob in the cluster. It would be unacceptably inefficient to decompose the entire cluster into blobs starting from scratch each time we must make a cloning decision. Instead, we use a new algorithm for cluster decomposition which allows us to incrementally decompose the cluster into blobs. At the end of an interval of  $n$  chemical shells of growth, we need only consider the effect on the cluster decomposition of the sites and bonds we have added to the cluster during the interval. The algorithm, based on the algorithm of Ref. [8] for determining the cluster backbone, works as follows:

- (i) During the growth of the cluster we identify “loop sites.” Loop sites are sites which are reached from two or more different growth sites simultaneously [8, 9].
- (ii) At the end of an interval of growth, we use the burning algorithm [8, 9] to walk back from each loop site toward the origin of the cluster. When we reach a state during the walk when only one site is burning, all sites traversed so far compose a blob. If during the walk we hit an existing blob,

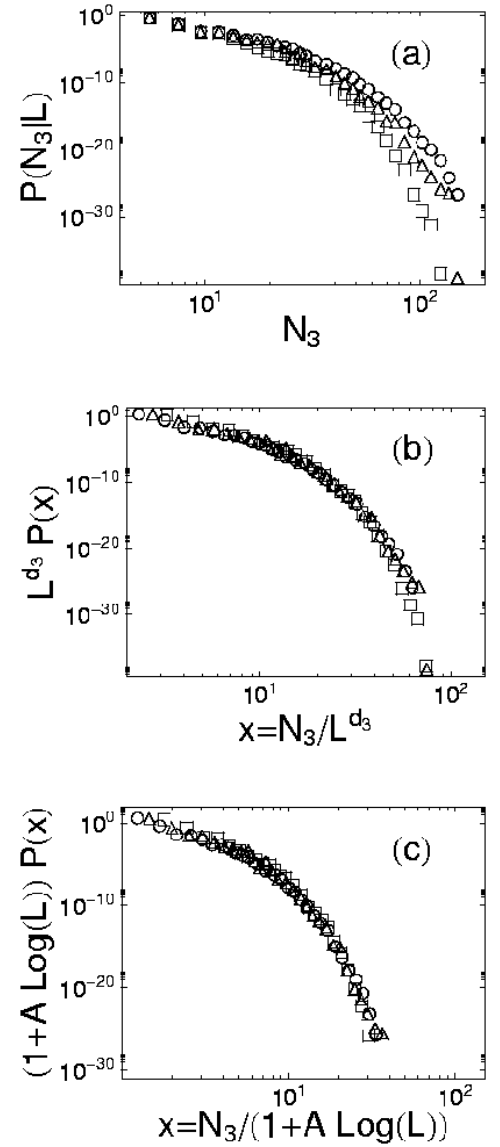


FIG. 4:  $P(N_3|L)$ , the distribution of 3-block mass  $N_3$  for (from left to right)  $L = 8, 16$ , and  $32$  for the case of six dimensions. (a) uncollapsed (b) collapsed using a value of  $d_3 = 0.25$ . (c) Collapsed assuming  $N_3 \sim 1 + A \log L$  with  $A = 1.0$ .

that blob is incorporated into the blob associated with the loop site from which the walk started.

- (iii) When we have exhausted all clones created at the end of an interval, we must restore the system to its state at the beginning of the interval. That is, we must:

- (a) destroy all blobs created,
- (b) separate any blobs which were merged, and
- (c) reduce any blobs which grew during the interval back to their size at the beginning of the

interval.

This is all accomplished by carefully maintaining the appropriate state information during the growth and cluster decomposition processes.

#### IV. RESULTS AND DISCUSSION

Using the methods described in the previous section, we generate percolation clusters on hypercubic lattices for 4d, 5d and 6d at their respective percolation thresholds [10, 11].

In Fig. 2a, we plot  $P(N_3|L)$ , the distribution of 3-block mass  $N_3$  in a system of size  $L$  for various  $L$  for  $d = 4$ . In Fig. 2b, we plot the same distributions collapsed using the estimated value  $d_3 = 0.7$  which, visually, yields the best collapse. We show analogous plots for  $d = 5$  and  $d = 6$  Figs. 3 and 4. Based on the value of  $d_3$  which yields the best collapse, we estimate

$$d_3 = \begin{cases} 0.7 \pm 0.2 & (4d) \\ 0.5 \pm 0.2 & (5d). \end{cases} \quad (1)$$

If we fit our results for 6d with a power law, then we find the best collapse is obtained for  $d_3 = 0.25 \pm 0.2$ . However it is difficult to numerically distinguish between power-law scaling with a small exponent and logarithmic scaling. Hence in Fig. 4c we also collapse the distributions for 6d assuming  $d_3 = 0$  with logarithmic corrections

to scaling

$$N_3 \sim 1 + A \log L \quad (6d) \quad (2)$$

with  $A = 1.0$ . The quality of the collapses for power law scaling and logarithmic scaling seem to be comparable; however, the facts that  $d_3 = 0$  for the Cayley tree and that logarithmic corrections to scaling are common at the upper critical dimension favor the conclusion that  $d_3 = 0$  for  $d = 6$ .

Finally, we make two observations:

(i) We note that the behavior of  $d_3$  with dimension is qualitatively the opposite of the behavior of  $d_2$ , the blob fractal dimension, in the following sense:  $d_2$  increases significantly between  $d = 2$  and  $d = 3$  but increases very slowly between  $d = 3$  and  $d = 6$  [2, 13, 14, 15], while  $d_3$  is slowly decreasing between  $d = 2$  and  $d = 3$  but then decreases significantly between  $d = 3$  and  $d = 6$ .

(ii) Since  $k = 0$  corresponds to the entire system, which scales as  $L^d$ , we note that for  $k = 0, 1, 2$ , and  $3$ , the fractal dimensions  $d_k$  for 6d are 6, 4, 2, and 0 respectively, that is, a series of decreasing consecutive even integers.

#### Acknowledgments

We thank Don Baker, Sergey Buldyrev, Shlomo Havlin, Greg Huber, and Sameet Sreenivasan for helpful discussions and NSF, BP, and Intevp for support.

- 
- [1] D. Ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press, Cambridge, 2000).
  - [2] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, Philadelphia, 1994).
  - [3] A. Bunde and S. Havlin, in *Fractals and Disordered Systems*, edited by A. Bunde and S. Havlin (Springer-Verlag, New York, 1996).
  - [4] G. Paul and H. E. Stanley, Phys. Rev. E **65**, 056126 (2002).
  - [5] J. L. Jacobsen and P. Zinn-Justin, cond-mat/0207063.
  - [6] P. Grassberger, Computer Physics Communications **147**, 64 (2002).
  - [7] P. L. Leath, Phys. Rev. B **14**, 5046 (1976).
  - [8] M. Porto, A. Bunde, S. Havlin, and H. E. Roman, Phys. Rev. E **56**, 1667 (1997).
  - [9] H. J. Herrmann and H. E. Stanley, Phys. Rev. Lett. **53**, 1121 (1984); H. J. Herrmann, D. C. Hong, and H. E. Stanley, J. Phys. A **17**, L261 (1984).
  - [10] G. Paul, R. M. Ziff, and H. E. Stanley, Phys. Rev. E **64**, 026115 (2001).
  - [11] P. Grassberger, cond-mat/0202144.
  - [12] J. L. Jacobsen and P. Zinn-Justin, J. Phys. A-Math. Gen. **35**, 2131 (2002).
  - [13] P. Grassberger, Physica A **262**, 251 (1999).
  - [14] C. Moukarzel, Int. J. Mod. Phys. C **8**, 887 (1998).
  - [15] H. K. Janssen and O. Stenull, Phys. Rev. E **61**, 4821 (2000).